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Restrictions on the Coefficients of Approximating Polynomials*

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1. INTRODUCTION

There has been some recent interest on possible restrictions on the coefficients of approximating polynomials [5–6]. Specifically, the problem being investigated is stated as follows: Let $S = \{A_k\}$ be a sequence of nonnegative real numbers. Let $H_S = \{p(x) \mid p(x) = \sum a_k x^k \text{ and } |a_k| \leqslant A_k{}^k\}$. Let C_0 be the set of all continuous real-valued functions f on [0, 1] for which $f(0) = 0$. What are necessary and sufficient conditions on the sequence S so that H_S is dense in C_0 in the uniform norm? Stafney [6] proved the following:

THEOREM A. *If* $\lim_{k\to\infty} A_k = +\infty$, *then* H_S *is dense in* C_0.

THEOREM B. *If* $\overline{\lim}_{k\to\infty} A_k < +\infty$, *then* H_S *is not dense in* C_0.

The present author improved somewhat on Theorem A with the following [5]:

THEOREM C. *If for each* $0 < \delta < 1$ *and each* $M > 0$ *there exist arbitrarily large* $N = N(\delta, M)$ *for which* $A_k \geqslant M$ *if* $N\delta \leqslant k \leqslant N$, *then* H_S *is dense in* C_0.

This paper improves significantly on all three of these theorems. The notion of generalized Bernstein polynomial [4, p. 47] plays a critical role in some of the proofs.

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2. The Main Theorems

A divergent series of positive terms $\sum_{k=1}^{\infty} 1/\alpha_k$ is said to have *property P* if for any $N > 0$ there is a $\delta > 0$ such that $\sum_{j=k}^{n} 1/\alpha_j > N$ whenever $\alpha_k/\alpha_n < \delta$.

With this notation we now state

THEOREM 1. *If the sequence $S = \{A_k\}$ contains a subsequence $\{A_{\alpha_k}\}$ for which $\lim_{k\to\infty} A_{\alpha_k} = +\infty$ and for which the series $\sum_{j=1}^{\infty} 1/\alpha_j$ has property P, then H_S is dense in C_0.*

THEOREM 2. *If $\{A_k\}$ is bounded on the complement of a set of positive integers $\{\alpha_1, \alpha_2, ...\}(\alpha_1 < \alpha_2 < \cdots)$ such that $\sum_{k=1}^{\infty} 1/\alpha_k < \infty$, then H_S is not dense in C_0. In fact, any function f in the uniform closure of H_S must be analytic on some subinterval of* [0, 1].

We note without proof that the hypothesis in Theorem 2 is equivalent to the following:

Any subsequence $\{A_{\alpha_k}\}$ of the sequence $\{A_k\}$ for which

$$\lim_{k\to\infty} A_{\alpha_k} = +\infty \text{ satisfies } \sum_{k=1}^{\infty} \frac{1}{\alpha_k} < \infty.$$

The proofs of these theorems require several lemmas.

If $\{\alpha_k\}$ is a sequence of integers satisfying

$$\lim_{k\to\infty} \alpha_k = +\infty; \tag{1}$$

$$\begin{aligned} &\alpha_1 = 1, \\ &0 = \alpha_0 < \alpha_1 < \alpha_2 < \alpha_3 < \cdots < \alpha_n < \alpha_{n+1} < \cdots, \end{aligned} \tag{2}$$

$$\sum_{k=1}^{\infty} \frac{1}{\alpha_k} = +\infty, \tag{3}$$

then define

$$P_{n_k}(x) = (-1)^{n-k}\, \alpha_{k+1} \cdots \alpha_n \sum_{j=k}^{n} x^{\alpha_j}/w_k'(\alpha_j),$$

where $k \leqslant n$ are non-negative integers and where

$$w_k(x) = (x - \alpha_k)(x - \alpha_{k+1}) \cdots (x - \alpha_n).$$

That is

$$p_{nk}(x) = (-1)^{n-k}\,\alpha_{k+1}\cdots\alpha_n \times \sum_{j=k}^{n} x^{\alpha_j}/[(\alpha_j-\alpha_k)\cdots(\alpha_j-\alpha_{j-1})(\alpha_j-\alpha_{j+1})\cdots(\alpha_j-\alpha_n)].$$

Let $a_{nk} = (1-(1/\alpha_{k+1}))\cdots(1-1/\alpha_n)$. If $f\in C[0,1]$, define

$$B_n{}^f(x) = \sum_{k=0}^{n} f(a_{nk})\,p_{nk}(x).$$

It is known that for any $f\in C\,[0,1]$. $B_n{}^f \to f$ uniformly on $[0, 1]$ [4, p. 47]. Now let

$$b_{nk} = \sum_{j=k}^{n} 1/[(\alpha_j-\alpha_k)\cdots(\alpha_j-\alpha_{j-1})\cdot(\alpha_n-\alpha_j)\cdots(\alpha_{j+1}-\alpha_j)].$$

We now state the following:

LEMMA 1. *If $f\in C\,[0,1]$ and if l is a nonnegative integer define $p_{nl}(x) = \sum_{k=l}^{n} f(a_{nk})\,p_{nk}(x) = \sum_{k=l}^{n} c_{kl}x^l$. Then there is a sequence $\{n_i\}$ of positive integers and a positive constant K (both independent of l) for which*

$$\sum_{k=l}^{n_i} |\,c_{kl}\,| \leqslant \|f\|\,K^{\alpha_{n_i}}, \qquad n_i \geqslant l. \tag{4}$$

Proof. We first prove that

$$\sum_{k=0}^{n} \alpha_{k+1}\cdots\alpha_n b_{nk} \leqslant K^{\alpha_n} \tag{5}$$

for some positive K and infinitely many n. First, note that if $r > s$ then $\alpha_r - \alpha_s \geqslant r - s$.

$$\text{(i)} \qquad b_{nk} \leqslant \sum_{j=k}^{n} \frac{1}{(j-k)!\,(n-j)!} = \frac{2^{n-k}}{(n-k)!}\,.$$

Also note that

$$\text{(ii)} \qquad \alpha_n\cdots\alpha_{k+1} = (n!/k!)\,\epsilon_n\cdots\epsilon_{k+1} < (n!/k!)\,\epsilon_1\cdots\epsilon_n\,.$$

Here we let $\alpha_j = j\epsilon_j$ where, of course, $\epsilon_j \geqslant 1$ $j = 0, 1,\ldots$. Combining (i) and (ii) we have

$$\text{(iii)} \qquad \sum_{k=0}^{n} \alpha_n\cdots\alpha_{k+1}b_{nk} \leqslant \sum_{k=0}^{n} \binom{n}{k} 2^{n-k}\epsilon_1\cdots\epsilon_n = \epsilon_1\cdots\epsilon_n 3^n.$$

We now have two cases:

Case 1. There is a number $M > 0$ such that $\epsilon_j \leqslant M$ all j. In this case we see from (iii) that

$$\sum_{k=0}^{n} \alpha_n \cdots \alpha_{k+1} b_{nk} \leqslant (3M)^n \qquad \text{for all } n.$$

Case 2. $\lim_{j\to\infty} \epsilon_{n_j} = +\infty$ for some subsequence $\{\epsilon_{n_j}\}$.
We may assume without loss of generality that $\epsilon_{n_j} \geqslant \epsilon_k$ for each $k \leqslant n_j$. Then from (iii) we have

$$\sum_{k=0}^{n_j} \alpha_{n_j} \cdots \alpha_{k+1} b_{n_j k} \leqslant (\epsilon_{n_j})^{n_j} 3^{n_j} \leqslant (2^{\epsilon_{n_j}})^{n_j} 3^{n_j} \leqslant 6^{\alpha_{n_j}} \qquad \text{for each } j.$$

This concludes the proof of (5).

Equation (4) now follows from (5) and the inequality

$$\sum_{k=l}^{n} | c_k | \leqslant \|f\| \sum_{k=l}^{n} \alpha_{k+1} \cdots \alpha_n b_{nk} \leqslant \|f\| \sum_{k=0}^{n} \alpha_{k+1} \cdots \alpha_n b_{nk} .$$

LEMMA 2. *Let* $0 < \alpha_1 < \alpha_2 < \cdots$ *be positive integers. Then there is a* $\delta > 0$ *such that for all* k *and* n *the following inequality is true*:

$$\frac{\delta}{\exp[(1/\alpha_k) + \cdots + (1/\alpha_n)]} \leqslant (1 - 1/\alpha_k) \cdots (1 - 1/\alpha_n)$$
$$\leqslant [(1/\alpha_k) + \cdots + (1/\alpha_n)]^{-1}. \qquad (6)$$

Proof. The right half of this inequality is well-known and the proof is omitted. We prove the left half. Since the α_i are integers we know that $\sum_{i=1}^{\infty} 1/\alpha_i^2$ converges. Hence,

$$\prod_{j=1}^{\infty} \left(1 - \frac{1}{\alpha_j^2}\right) = \delta > 0.$$

So, we have

$$\delta = \prod_{j=1}^{\infty} \left(1 - \frac{1}{\alpha_j^2}\right) < \prod_{j=k}^{n} \left(1 - \frac{1}{\alpha_j^2}\right) = \prod_{j=k}^{n} \left(1 - \frac{1}{\alpha_j}\right) \cdot \prod_{i=k}^{n} \left(1 + \frac{1}{\alpha_i}\right).$$

And so,

$$\frac{\delta}{\prod_{i=k}^{n} (1 + 1/\alpha_i)} \leqslant \prod_{j=k}^{n} \left(1 - \frac{1}{\alpha_j}\right).$$

(6) now follows from this and the known inequality

$$\prod_{i=k}^{n} (1 + 1/\alpha_i) \leqslant \exp[(1/\alpha_k) + \cdots + (1/\alpha_n)].$$

Proof of Theorem 1. Let $f \in C_0$ and let $\epsilon > 0$ be given. Choose $\delta_0 > 0$ such that $|f(x)| < \epsilon/2$ if $0 < x < \delta_0$. Let $\{n_j\}$ be the sequence of integers for which Lemma 1 is true, and so that $n_1 < n_2 < \cdots$.

Choose integer $M > 0$ such that $\|f - B_{n_j}^f\| < \epsilon/2$, whenever $j > M$. By property P, choose $\delta_1 > 0$ such that $(\alpha_k/\alpha_n) < \delta_1$ implies $\sum_{j=k+1}^{n} 1/\alpha_j > 1/\delta_0$. Then by (6) we have

$$a_{nk} = (1 - 1/\alpha_{k+1}) \cdots (1 - 1/\alpha_n) < \delta_0 \qquad \text{if} \quad (\alpha_k/\alpha_n) < \delta_1.$$

From this we see that

$$f(a_{nk}) < \epsilon/2 \qquad \text{if} \quad \alpha_k/\alpha_n < \delta_1.$$

For each j, define

$$Q_j(x) = B_{n_j}^f(x) - \sum_{\alpha_k/\alpha_{n_j} < \delta_1} f(a_{n_jk})\, p_{n_jk}(x).$$

If $j > M$ then

$$|f(x) - Q_j(x)| \leqslant |f(x) - B_{n_j}^f(x)| + \sum_{\alpha_k/\alpha_{n_j} < \delta_1} |f(a_{n_jk})|\, |p_{n_jk}(x)|$$

$$\leqslant \frac{\epsilon}{2} + \frac{\epsilon}{2} \sum_{\alpha_k/\alpha_{n_j} < \delta_1} p_{n_jk}(x) < \epsilon.$$

It should be noted that in the above estimate we used the fact that $p_{nk}(x) \geqslant 0$ and $\sum_{k=0}^{n} p_{nk}(x) = 1$ [4, p. 46].

Let

$$Q_j(x) = \sum_{k=l}^{n_j} C_k x^{\alpha_k}.$$

Then by Lemma 1 we have

$$\sum_{k=l}^{n_j} |C_k| \leqslant \|f\|\, K^{\alpha_{n_j}}.$$

But $k \geqslant l$ if and only if $\alpha_k/\alpha_{n_j} \geqslant \delta_1$. (i.e., $\alpha_{n_j} \leqslant \alpha_k/\delta_1$).

Hence,

$$|C_k| \leqslant \|f\|\, K^{\alpha_{n_j}}$$

$$\leqslant \|f\|\, K^{\frac{\alpha_k}{\delta_1}} = [\|f\|^{1/\alpha_k} K^{1/\delta_1}]^{\alpha_k}, \qquad k = 1,\ldots, n.$$

We now choose j so large that

$$A_{\alpha_k} > \|f\|^{1/\alpha_k} K^{1/\delta_1}$$

for all k for which $\alpha_k/\alpha_{n_j} \geqslant \delta_1$.

Then we have $|c_k| \leqslant A_{\alpha_k}^{\alpha_k}$ for $\alpha_k/\alpha_{n_j} \geqslant \delta_1$ and $c_k = 0$ for $\alpha_k/\alpha_{n_j} < \delta_1$. This completes the proof of Theorem 1.

The proof of Theorem 2 uses two lemmas.

The following lemma is well known and the proof is omitted [1, p. 171].

LEMMA 3. *A family $\mathscr{F}$ of functions in a region Ω is normal if the functions $f \in \mathscr{F}$ are uniformly bounded on every compact subset of Ω.*

(The definition of normal family is the same as that in [1, p. 168] and is repeated here for the reader's convenience.)

A family $\mathscr{F}$ of functions f, defined in a region Ω, is said to be *normal* if every sequence $\{f_n\}$ of functions $f_n \in \mathscr{F}$ contains a subsequence $\{f_{n_k}\}$ which either converges uniformly or tends uniformly to ∞ on every compact subset of Ω. The proof of the following lemma is given in [2, p. 7] and is omitted here.

LEMMA 4 (Clarkson, Erdos). *If n_i is a sequence of positive integers which satisfy $\sum (1/n_i) < +\infty$, and if f is uniformly approximable on* [0, 1] *by polynomials involving only powers x^{n_i}, then f is analytic on* [0, 1).

Proof of Theorem 2. Suppose f is in the closure of H_S. Then there is a sequence $\{p_n\}$ of polynomials from H_S which converges to f uniformly on [0, 1]. We see that if $p_n(x) = \sum_{k=0}^{n} a_{nk}x^k$ then $|a_{nk}| \leqslant A_k{}^k$ for all n, k. Let $\{\beta_j\}$ be the sequence of integers consisting of the complement of $\{\alpha_k\}$.

For each positive integer n we now define

$$q_n(x) = \sum_{\alpha_k \leqslant n} a_{n,\alpha_k} x^{\alpha_k} \qquad \text{and} \qquad r_n(x) = \sum_{\beta_j \leqslant n} a_{n,\beta_j} x^{\beta_j}.$$

With these definitions we see that $p_n(x) = q_n(x) + r_n(x)$ $n = 1, 2, \ldots$. But now we have the inequality

$$|r_n(x)| \leqslant \sum_{\beta_j \leqslant n} |a_{n,\beta_j}| x^{\beta_j} \leqslant \sum_{\beta_j \leqslant n} A_{\beta_j}^{\beta_j} x^{\beta_j} \leqslant \sum_{\beta_j \leqslant n} c^{\beta_j} x^{\beta_j},$$

(c a constant independent of j and n).

So, on the interval [0, $1/4c$) we have $|r_n(x)| \leqslant \sum (1/4)^{\beta_j} \leqslant \sum_{j=0}^{\infty} (1/4)^j = 4/3$. In fact, if we consider r_n as a function of a complex variable we have $|r_n(z)| \leqslant 4/3$ for all z satisfying $|z| < 1/4c$. Hence, $r_n(z)$ is uniformly bounded on every compact subset of the circle $|z| < 1/4c$.

So, by Lemma 3, the sequence $\{r_n\}$, $n = 1, 2, \ldots$, is a normal family. Hence, there must be a subsequence $\{r_{n_k}\}$ which converges uniformly to a function h on every compact subset of the circle $|z| < 1/4c$. So, h is analytic on $[0, 1/4c)$. Now consider the sequence $q_{n_k}(x) = p_{n_k}(x) - r_{n_k}(x)$. We see that $q_{n_k}(x)$ tends uniformly to $f - h$ on $[0, 1/8c]$. Lemma 4 clearly implies, then, that $f - h$ is analytic on $[0, 1/8c)$. But then f must be analytic on $[0, 1/8c)$. This completes the proof of Theorem 2.

3. Remarks

(a) We see that the "gap" between Theorem 1 and Theorem 2 is the rapidity with which the series $\sum (1/\alpha_j)$ diverges. One might argue that this difficulty might have been avoided by considering partial products instead of partial sums. Lemma 4 however, clearly shows that these two approaches are equivalent.

(b) The motivation for the proof of Theorem 1 is the same as the general approach used in [5]. The only difference is that this paper uses the "generalized" Bernstein polynomials instead of the usual ones. The main difficulty here is with the points a_{nk} which do not, in general, behave like the points k/n in the usual Bernstein polynomial.

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